Theorem III. (*Derivative of scalar product of two vector functions*)

If $\vec{u}(t)$ and $\vec{v}(t)$ be two differentiable functions of the scalar t, to show that

$$\frac{d}{dt}(\vec{u}\cdot\vec{v})=\vec{u}\cdot\frac{d\vec{v}}{dt}+\frac{d\vec{u}}{dt}\cdot\vec{v}.$$

(M. U. 1987; B. U. '86; P. U. '86)

Proof. Let $$w=\vec{u}\cdot\vec{v}. \quad \ldots (1)$$

Let $\delta\vec{u}$ and $\delta\vec{v}$ be the small increments in $\vec{u}$ and $\vec{v}$ respectively and δw be the corresponding small increment in w.

Then $w+\delta w=(\vec{u}+\delta\vec{u}).(\vec{v}+\delta\vec{v})$

or $$w+\delta w=\vec{u}\cdot\vec{v}+\vec{u}\cdot\delta\vec{v}+\delta\vec{u}\cdot\vec{v}+\delta\vec{u}\cdot\delta\vec{v}. \quad \ldots (2)$$

Subtracting the corresponding sides of (1) from (2), we get

$$w+\delta w-w=\vec{u}\cdot\vec{v}+\vec{u}\cdot\delta\vec{v}+\delta\vec{u}\cdot\vec{v}+\delta\vec{u}\cdot\delta\vec{v}-\vec{u}\cdot\vec{v}$$

or $\delta w=\vec{u}\cdot\delta\vec{v}+\delta\vec{u}\cdot\vec{v}+\delta\vec{u}\cdot\delta\vec{v}$.

Dividing both sides by δt, we get

$$\frac{\delta w}{\delta t}=\vec{u}\cdot\frac{\delta\vec{v}}{\delta t}+\frac{\delta\vec{u}}{\delta t}\cdot\vec{v}+\frac{\delta\vec{u}}{\delta t}\cdot\frac{\delta\vec{v}}{\delta t}\cdot\delta t.$$

Now proceeding to limits as $\delta t\to 0$, we get

$$\underset{\delta t\to 0}{\mathrm{Lt}}\frac{\delta w}{\delta t}=\vec{u}\cdot\underset{\delta t\to 0}{\mathrm{Lt}}\frac{\delta\vec{v}}{\delta t}+\underset{\delta t\to 0}{\mathrm{Lt}}\frac{\delta\vec{u}}{\delta t}\cdot\vec{v}+\underset{\delta t\to 0}{\mathrm{Lt}}\frac{\delta\vec{u}}{\delta t}\cdot\underset{\delta t\to 0}{\mathrm{Lt}}\frac{\delta\vec{u}}{\delta t}\cdot\underset{\delta t\to 0}{\mathrm{Lt}}\,\delta t$$

or $$\frac{dw}{dt}=\vec{u}\cdot\frac{d\vec{v}}{dt}+\frac{d\vec{u}}{dt}\cdot\vec{v}+\frac{d\vec{u}}{dt}\cdot\frac{d\vec{v}}{dt}\cdot 0$$

or $$\frac{d}{dt}(\vec{u}\cdot\vec{v})=\vec{u}\cdot\frac{d\vec{v}}{dt}+\frac{d\vec{u}}{dt}\cdot\vec{v}.$$

Note. (i) We know that the scalar product of two vectors is commutative.

$$\therefore\quad \frac{d\vec{u}}{dt}\cdot\vec{v}=\vec{v}\cdot\frac{d\vec{u}}{dt}.$$

Hence $$\frac{d}{dt}(\vec{u}\cdot\vec{v})=\vec{u}\cdot\frac{d\vec{v}}{dt}+\vec{v}\cdot\frac{d\vec{u}}{dt}.$$

(ii) Taking $\vec{v}=\vec{u}$, we have

$$\frac{d}{dt}(\vec{u}\cdot\vec{u})=\vec{u}\cdot\frac{d\vec{u}}{dt}+\frac{d\vec{u}}{dt}\cdot\vec{u}$$

or $$\frac{d}{dt}(\vec{u}^{\,2})=2\vec{u}\cdot\frac{d\vec{u}}{dt}.$$

Theorem IV. (*Derivative of vector product of two vector functions*)

(Mith. U. 1986)

If $\vec{u}(t)$ *and* $\vec{v}(t)$ *be two differential functions of the scalar t, to show that*

$$\frac{d}{dt}(\vec{u} \times \vec{v}) = \vec{u} \times \frac{d\vec{v}}{dt} + \frac{d\vec{u}}{dt} \times \vec{v}.$$

(M. U. 1987; B. U. '86; Mith. U. '85)

Proof. Let $\vec{w} = \vec{u} \times \vec{v}$. .. (1)

Let $\delta \vec{u}$ and $\delta \vec{v}$ be the small increments in $\vec{u}$ and $\vec{v}$ respectively and $\delta \vec{w}$ be the corresponding small increment in $\vec{w}$.

Then $\vec{w} + \delta \vec{w} = (\vec{u} + \delta \vec{u}) \times (\vec{v} + \delta \vec{v})$

or $\vec{w} + \delta \vec{w} = \vec{u} \times \vec{v} + \vec{u} \times \delta \vec{u} + \delta \vec{u} \times \vec{v} + \delta \vec{u} \times \delta \vec{v}$. .. (2)

Subtracting the corresponding sides of (1) from (2), we get

$$\vec{w} + \delta \vec{w} - \vec{w} = \vec{u} \times \vec{v} + \vec{u} \times \delta \vec{v} + \delta \vec{u} \times \vec{v} + \delta \vec{u} \times \delta \vec{v} - \vec{u} \times \vec{v}$$

or $\delta \vec{w} = \vec{u} \times \delta \vec{v} + \delta \vec{u} \times \vec{v} + \delta \vec{u} \times \delta \vec{v}$.

Dividing both sides by δt, we get

$$\frac{\delta \vec{w}}{\delta t} = \vec{u} \times \frac{\delta \vec{v}}{\delta t} + \frac{\delta \vec{u}}{\delta t} \times \vec{v} + \frac{\delta \vec{u}}{\delta t} \times \frac{\delta \vec{v}}{\delta t} . \delta t.$$

Now proceeding to limits as $\delta t \to 0$, we get

$$\underset{\delta t \to 0}{\text{Lt}} \frac{\delta \vec{w}}{\delta t} = \vec{u} \times \underset{\delta t \to 0}{\text{Lt}} \frac{\delta \vec{v}}{\delta t} + \underset{\delta t \to 0}{\text{Lt}} \frac{\delta \vec{u}}{\delta t} \times \vec{v} + \underset{\delta t \to 0}{\text{Lt}} \frac{\delta \vec{u}}{\delta t} \times \underset{\delta t \to 0}{\text{Lt}} \frac{\delta \vec{v}}{\delta t} . \underset{\delta t \to 0}{\text{Lt}} \delta t$$

or $$\frac{d\vec{w}}{dt} = \vec{u} \times \frac{d\vec{v}}{dt} + \frac{d\vec{u}}{dt} \times \vec{v} + \frac{d\vec{u}}{dt} \times \frac{d\vec{v}}{dt} . 0$$

or $$\frac{d}{dt}(\vec{u} \times \vec{v}) = \vec{u} \times \frac{d\vec{v}}{dt} + \frac{d\vec{u}}{dt} \times \vec{v}.$$

Note. We know that the vector product of two vectors is not commutative. that is, anti-commutative.

$$\therefore \quad \frac{d\vec{u}}{dt} \times \vec{v} = -\vec{v} \times \frac{d\vec{u}}{dt}.$$

Hence the order of the factors in the formula

$$\frac{d}{dt}(\vec{u} \times \vec{v}) = \vec{u} \times \frac{d\vec{v}}{dt} + \frac{d\vec{u}}{dt} \times \vec{v}$$

must be strictly maintained.